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1997 J. Phys. A: Math. Gen. 30 L519

(http://iopscience.iop.org/0305-4470/30/15/007)

J. Phys. A: Math. Gen. **30** (1997) L519–L523. Printed in the UK PII: S0305-4470(97)79032-2

LETTER TO THE EDITOR

Models of passive and reactive tracer motion: an application of Ito calculus

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Received 24 February 1997, in final form 16 June 1997

Abstract. By means of Ito calculus it is possible to find, in a straightforward way, the analytical solutions to some equations related to the passive tracer transport problem in a velocity field that obeys the multidimensional Burgers equation and solutions to a simple model of reactive tracer motion.

1. Introduction

In a recent paper [1] Saichev and Woyczynski have obtained exact solutions in arbitrary dimensions of equations of hydrodynamic type related to the Burgers equation (see, e.g., [2–6]) for an irrotational velocity field with two models of coupled passive or reactive tracers. For the inhomogeneous Burgers equation and more general models of tracers, closed but less explicit solutions are obtained as path integrals.

The main idea of [1] is to reduce the system specified by the forced Burgers equation, together with the advection–diffusion–reaction equation, to a pair of coupled linear diffusion equations with variable coefficient which can be analytically solved by means of the Feynman–Kac formula (see, e.g., [6]).

They also show that the same methodology can be used to generate exact solutions of a nonlinear reaction–diffusion equation coupled with a Burgers-like velocity field also depending on the concentration.

The Feynman–Kac equation expresses the solution of a parabolic partial differential equation (PDE) without drift in terms of a conditional average over Brownian trajectories. A generalization of the Feynman–Kac formula when a drift is present is supplied by the Cameron–Martin–Girsanov formula [7]. This observation allows us to recover the results in [1] without using any auxiliary field.

2. The Girsanov formula

In this section a generalized version of the Girsanov formula is recalled and its relevance to parabolic PDEs is explained.

Theorem 1. Let $\boldsymbol{x}^{(1)}$, $\boldsymbol{x}^{(2)} \in \mathbb{R}^d$ be solutions on the interval $0 \leqslant t \leqslant T$ of the stochastic differential equations

$$\begin{aligned} \mathrm{d}\boldsymbol{x}^{(i)}(t) &= \boldsymbol{b}^{(i)}(\boldsymbol{x}^{(i)}(t), t)\,\mathrm{d}t + \sigma(\boldsymbol{x}^{(i)}(t), t)\,\mathrm{d}\boldsymbol{w}(t) \\ \boldsymbol{x}^{(i)}(0) &= \boldsymbol{x} \qquad i = 1, 2 \end{aligned} \tag{1}$$

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where $\boldsymbol{w}$ is a d-dimensional Brownian motion, $\boldsymbol{b}^{(i)}(\boldsymbol{x}, t)$ $(i = 1, 2)$ and $\sigma(t, \boldsymbol{x})$ are respectively Borel-measurable, $\mathbb{R}^d$-valued functions on $[0, \infty[\times\mathbb{R}^d$ and a Borel-measurable, $[0, \infty[\times\mathbb{R}^d \times \mathbb{R}^d$-valued function with bounded inverse $\forall \boldsymbol{x}, t$. If $\boldsymbol{b}^{(i)}(\boldsymbol{x}, t)$ $(i = 1, 2)$ and $\sigma(t, \boldsymbol{x})$ satisfy the assumptions of the existence and uniqueness theorem for the solutions of (1), then the probability measure μ_2 of $\boldsymbol{x}^{(2)}$ will be absolutely continuous with respect to μ_1 of $\boldsymbol{x}^{(1)}$ and

$$\begin{aligned} &\frac{\mathrm{d}\mu_2}{\mathrm{d}\mu_1}(\boldsymbol{x}^{(1)}(t)) = \mathrm{e}^{\mathcal{A}_t} \\ &\mathcal{A}_t = \int_0^t \boldsymbol{\alpha}(\boldsymbol{x}^{(1)}(s), s) \cdot \mathrm{d}\boldsymbol{w}(s) - \frac{1}{2}\int_0^t \|\boldsymbol{\alpha}(\boldsymbol{x}^{(1)}(s), s)\|^2 \, \mathrm{d}s \end{aligned} \tag{2}$$

where

$$\boldsymbol{\alpha}(\boldsymbol{x}, t) = \sigma^{-1}(\boldsymbol{x}, t)[\boldsymbol{b}^{(2)}(\boldsymbol{x}, t) - \boldsymbol{b}^{(1)}(\boldsymbol{x}, t)\}. \tag{3}$$

See [7, p 279] for proof and details.

A straightforward consequence is that for any reasonably smooth function $f_k(x)$, with k ranging from 1 to n and for all n-tuples $(t_1, \ldots, t_n)$ such that $0 \leqslant t_1 \leqslant \cdots \leqslant t_n \leqslant t$ we have

$$E^{\boldsymbol{x}}\left\{\prod_{k=1}^{n} f_k(\boldsymbol{x}^{(2)}(t_k))\right\} = E^{\boldsymbol{x}}\left\{\prod_{k=1}^{n} f_k(\boldsymbol{x}^{(1)}(t_k))\, \mathrm{e}^{\mathcal{A}_t}\right\}. \tag{4}$$

This result can be exploited in order to write the solutions of parabolic PDEs as path integrals on Wiener trajectories (see, e.g., [6]).

In the following sections the result (2) will be applied, disregarding the conditions on the drift field, in order to derive the announced results formally. The advantage of this approach is to supply a direct physical interpretation for the solutions in terms of stochastic trajectories.

3. Simple applications

As a first application, let us consider the d-dimensional homogeneous Burgers equation with rotation free initial condition:

$$\begin{aligned} &\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} = \nu \Delta \boldsymbol{v} \\ &\boldsymbol{v}(\boldsymbol{x}, 0) = \boldsymbol{\nabla}\Theta_0(\boldsymbol{x}). \end{aligned} \tag{5}$$

The physical meaning of (5) is that the velocity field is, on the average, constant along the trajectories generated by the stochastic differential equation:

$$\mathrm{d}\boldsymbol{x}(s) = -\boldsymbol{\nabla}\Theta(\boldsymbol{x}(s), t - s)\, \mathrm{d}s + \sqrt{2\nu}\, \mathrm{d}\boldsymbol{w}(s) \qquad \boldsymbol{x}(0) = \boldsymbol{x}. \tag{6}$$

If we introduce

$$\mathrm{d}\boldsymbol{z}(s) = \sqrt{2\nu}\, \mathrm{d}\boldsymbol{w}(s) \qquad \boldsymbol{z}(0) = \boldsymbol{x} \qquad \boldsymbol{z}(t) \stackrel{\text{law}}{=} \mathcal{N}(\boldsymbol{x}, 2\nu t) \tag{7}$$

we can exploit Girsanov's theorem and write

$$\boldsymbol{\nabla}\Theta(\boldsymbol{x}, t) = E^{\boldsymbol{x}}\{\boldsymbol{\nabla}_{z(t)}\Theta_0(\boldsymbol{z}(t))\, \mathrm{e}^{-\mathcal{Z}_t}\} \tag{8}$$

where

$$\mathcal{Z}_t = \frac{1}{\sqrt{2\nu}} \int_0^t \boldsymbol{\nabla}\Theta(\boldsymbol{z}(s), t - s) \cdot \mathrm{d}\boldsymbol{w}(s) + \frac{1}{4\nu}\int_0^t \|\boldsymbol{\nabla}\Theta(\boldsymbol{z}(s), t - s)\|^2 \, \mathrm{d}s. \tag{9}$$

We can eliminate the stochastic integral in (9) by means of

$$\mathrm{d}_s\Theta(\boldsymbol{z}(s), t-s) = \{-\partial_{t-s}\Theta(\boldsymbol{z}(s), t-s) + \nu\Delta\Theta(\boldsymbol{z}(s), t-s)\}\,\mathrm{d}s \\ +\sqrt{2\nu}\boldsymbol{\nabla}\Theta(\boldsymbol{z}(s), t-s)\cdot \mathrm{d}\boldsymbol{w}(s). \tag{10}$$

This substitution is useful since the potential satisfies

$$\partial_t\Theta + \tfrac{1}{2}\boldsymbol{\nabla}\Theta\cdot\boldsymbol{\nabla}\Theta = \nu\Delta\Theta \qquad \Theta(\boldsymbol{x}, 0) = \Theta_0(\boldsymbol{x}). \tag{11}$$

So we get to

$$\boldsymbol{\nabla}\Theta(\boldsymbol{x}, t)\exp\left[-\frac{\Theta(\boldsymbol{x}, t)}{2\nu}\right] = E^{\boldsymbol{x}}\left\{\boldsymbol{\nabla}_{z(t)}\Theta_0(\boldsymbol{z}(t))\exp\left[-\frac{\Theta_0(\boldsymbol{z}(t))}{2\nu}\right]\right\}. \tag{12}$$

Finally, we can derive the explicit expression for the velocity potential by means of a simple integration by parts and by exploiting the homogeneity of (7):

$$\Theta(\boldsymbol{x}, t) = -2\nu\ln E^{\boldsymbol{x}}\left\{\exp\left[-\frac{\Theta_0(\boldsymbol{z}(t))}{2\nu}\right]\right\} \tag{13}$$

which means

$$\Theta(\boldsymbol{x}, t) = -2\nu\ln\left[\int_{-\infty}^{\infty}\mathrm{e}^{-\Phi(\boldsymbol{x},\boldsymbol{y},t)/2\nu}\frac{\mathrm{d}^d y}{(4\pi\nu t)^{d/2}}\right] \tag{14}$$

where

$$\Phi(\boldsymbol{x}, \boldsymbol{y}, t) = \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{2t} + \Theta_0(\boldsymbol{y}). \tag{15}$$

If we consider a class of initial conditions such that

$$\frac{\Theta_0(\boldsymbol{x})}{\|\boldsymbol{x}\|^2}\to 0 \qquad \text{for } \|\boldsymbol{x}\|\uparrow\infty \tag{16}$$

then (14) is always well defined and, in the limit $t\downarrow 0$, it is consistent with the initial condition $\Theta(\boldsymbol{x}, 0) = \Theta_0(\boldsymbol{x})$. Therefore, we have recovered the already known result of the Hopf–Cole theory (see, e.g., [4, 5]) stating that the solution of the Burgers equation with irrotational initial condition is given in any dimension at arbitrary time t by

$$\boldsymbol{v}(\boldsymbol{x}, t) = \nabla\Theta(\boldsymbol{x}, t). \tag{17}$$

Let us now consider the system

$$\partial_t C + \boldsymbol{v}\cdot\boldsymbol{\nabla}C = \mu\Delta C + VC + g \qquad C(\boldsymbol{x}, 0) = C_0(\boldsymbol{x}) \tag{18}$$

where $\boldsymbol{v}$ is given by (5). The external drift V and the volume force g are functions of $\boldsymbol{x}$ and t; V, g and the initial data C_0 are smooth functions growing, as $\|\boldsymbol{x}\|$ goes to infinity, more slowly than $\|\boldsymbol{x}\|^2$. In the general case, this equation can be formally integrated as a path integral in the form (see, e.g., [6]):

$$C(x, t) = E^{\boldsymbol{x}}\left\{C_0(\boldsymbol{z}(t))\exp\left[-\mathcal{Z}_t + \int_0^t V(\boldsymbol{z}(s), t-s)\,\mathrm{d}s\right]\right\} \\ +E^{\boldsymbol{x}}\left\{\int_0^t g(\boldsymbol{z}(s), t-s)\exp\left[-\mathcal{Z}_s + \int_0^s V(\boldsymbol{z}(u), t-u)\,\mathrm{d}u\right]\mathrm{d}s\right\} \tag{19}$$

where $z(t)$ and $\mathcal{Z}_t$ are given by (7) and (9), respectively.

If $\mu = \nu$ and $V = g = 0$ the situation is more simple and the solution can be easily expressed as an ordinary integral:

$$C(\boldsymbol{x}, t) = E^{\boldsymbol{x}}\{C_0(\boldsymbol{z}(t))\,\mathrm{e}^{-\mathcal{Z}_t}\}. \tag{20}$$

It is now sufficient to proceed, as before, to the elimination of the stochastic integral in (9) by means of (10) and to use (11) to arrive at

$$C(\boldsymbol{x},t)=\exp\left[\frac{\Theta(\boldsymbol{x},t)}{2\nu}\right]\int_{-\infty}^{\infty}C_0(\boldsymbol{y})\exp\left[-\frac{\Phi(\boldsymbol{x},\boldsymbol{y},t)}{2\nu}\right]\frac{\mathrm{d}^d y}{(4\pi\nu t)^{d/2}}\tag{21}$$

or, in more explicit terms,

$$C(\boldsymbol{x},t)=\frac{\int_{-\infty}^{\infty}C_0(\boldsymbol{y})\,\mathrm{e}^{-\Phi(\boldsymbol{x},\boldsymbol{y},t)/2\nu}\,\mathrm{d}^d y}{\int_{-\infty}^{\infty}\mathrm{e}^{-\Phi(\boldsymbol{x},\boldsymbol{y},t)/2\nu}\,\mathrm{d}^d y}\tag{22}$$

which is the first result of Saichev and Woyczynski.

If the diffusion coefficient in (18) is $\mu\neq\nu$, the velocity and the concentration fields are, on average, constant along different stochastic trajectories, so we can no longer use (11) to reduce the path integral to a finite-dimensional one. However, we can formally write

$$\begin{aligned}C(\boldsymbol{x},t)&=\mathrm{e}^{\Theta(\boldsymbol{x},t)/2\mu}E^{\boldsymbol{x}}\left\{C_0(\boldsymbol{z}'(t))\exp\left[-\frac{\Theta_0(\boldsymbol{z}'(t))}{2\mu}-\frac{(\nu-\mu)}{2\mu}\int_0^t\Delta\Theta(\boldsymbol{z}'(s),t-s)\,\mathrm{d}s\right]\right\}\\ \boldsymbol{z}'(t)&\overset{\text{law}}{=}\mathcal{N}(\boldsymbol{x},2\mu t)\end{aligned}\tag{23}$$

where $\Theta(\boldsymbol{x},t)$ is given by (14). The expression (23) holds true as far it is well defined (non-divergent).

4. A reaction–diffusion model

Another possible simple application of the Girsanov theorem is the solution of the following reaction–diffusion model considered in [1]:

$$\partial_t\boldsymbol{v}+\boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v}=\nu\Delta\boldsymbol{v}+2\nu k\boldsymbol{\nabla}C\qquad \boldsymbol{v}(\boldsymbol{x},0)=\boldsymbol{\nabla}\Theta_0(\boldsymbol{x})\tag{24}$$

$$\partial_t C+\boldsymbol{v}\cdot\boldsymbol{\nabla}C=\nu\Delta C+kC^2\qquad C(\boldsymbol{x},0)=C_0(\boldsymbol{x})\tag{25}$$

where k is a constant.

Let us consider the stochastic differential

$$\begin{aligned}&\mathrm{d}_s\left\{\exp\left[k\int_0^s C(\boldsymbol{x}(u),t-u)\,\mathrm{d}u\right]C(\boldsymbol{x}(s),t-s)\right\}\\ &\quad=\exp\left[k\int_0^s C(\boldsymbol{x}(u),t-u)\,\mathrm{d}u\right]\{kC^2(\boldsymbol{x}(s),t-s)\,\mathrm{d}s+\mathrm{d}_s C(\boldsymbol{x}(s),t-s)\}\end{aligned}\tag{26}$$

where

$$\mathrm{d}\boldsymbol{x}(s)=-\boldsymbol{\nabla}\Theta(\boldsymbol{x}(s),t-s)\,\mathrm{d}s+\sqrt{2\nu}\,\mathrm{d}\boldsymbol{w}(s).\tag{27}$$

If we take the expectation value of (26) and equation (24) is satisfied, then it is easy to see that

$$C(\boldsymbol{x},t)=E^{\boldsymbol{x}}\left\{C_0(\boldsymbol{z}(t))\exp\left[-\mathcal{Z}_t+k\int_0^t C(\boldsymbol{z}(s),t-s)\,\mathrm{d}s\right]\right\}\tag{28}$$

where $z(t)$ and $\mathcal{Z}_t$ are given by (7) and (9), respectively.

Again we can use (10) and obtain an expression of the form (21) where, now, $\Theta(\boldsymbol{x},t)$ is given by solving (24). This can be done by observing that, in terms of the potential, it has, according to the notation in [1], the form

$$\partial_t\Theta+\tfrac{1}{2}\boldsymbol{\nabla}\Theta\cdot\boldsymbol{\nabla}\Theta=\nu\Delta\Theta+2\nu k\,\mathrm{e}^{\Theta/2\nu}a_0.\tag{29}$$

Here we have

$$a_0(\boldsymbol{x}, t) = \int_{-\infty}^{\infty} C_0(\boldsymbol{y})\, \mathrm{e}^{-\Phi(\boldsymbol{x},\boldsymbol{y},t)/2\nu} \frac{\mathrm{d}^d y}{(4\pi\nu t)^{d/2}}. \tag{30}$$

and $\Phi(\boldsymbol{x}, \boldsymbol{y}, t)$ is defined by (15).

The Ito calculus suggests that equation (29) can be rewritten by means of the stochastic process (7) as

$$E^{\boldsymbol{x}}\left\{\mathrm{d}_s \exp\left[-\frac{\Theta(\boldsymbol{z}(s), t-s)}{2\nu}\right]\right\} = kE^{\boldsymbol{x}}\{a_0(\boldsymbol{z}(s), t-s)\} \tag{31}$$

which implies

$$\Theta(\boldsymbol{x}, t) = -2\nu \ln E^{\boldsymbol{x}}\left\{\mathrm{e}^{-\Theta(\boldsymbol{z}(t),0)/2\nu} - kta_0(\boldsymbol{z}(t), 0) + \int_0^t s\, \mathrm{d}_s a_0(\boldsymbol{z}(s), t-s)\, \mathrm{d}s\right\}. \tag{32}$$

Since $a_0(\boldsymbol{x}, t)$ is solution of the heat equation we have

$$E^{\boldsymbol{x}}\left\{\int_0^t s\, \mathrm{d}_s a_0(\boldsymbol{z}(s), t-s)\, \mathrm{d}s\right\} = 0. \tag{33}$$

Therefore, we can conclude that the velocity potential is now

$$\Theta(\boldsymbol{x}, t) = -2\nu \ln[b_0(\boldsymbol{x}, t) - kta_0(\boldsymbol{x}, t)] \tag{34}$$

with

$$b_0(\boldsymbol{x}, t) = \int_{-\infty}^{\infty} \mathrm{e}^{-\Phi(\boldsymbol{x},\boldsymbol{y},t)/2\nu} \frac{\mathrm{d}^d y}{(4\pi\nu t)^{d/2}} \tag{35}$$

while the reactive tracer is

$$C(\boldsymbol{x}, t) = \frac{a_0(\boldsymbol{x}, t)}{b_0(\boldsymbol{x}, t) - kta_0(\boldsymbol{x}, t)} \tag{36}$$

which is the second result of Saichev and Woyczynski.

5. Conclusion

By means of Ito calculus is it possible to obtain the result reported by Saichev and Woyczynski in ([1]) in a natural and straightforward way.

This work benefited from sharp observations made by Erik Aurell. I am deeply grateful to him and to all the other friends in Sweden for their warm hospitality and good time spent together. This work was supported by a ESF/TAO grant for the year 1996.

After the submission of the paper I received a copy of the work by Garbaczewski *et al* [8] where the Burgers equation is investigated in the more general Schrödinger interpolation framework.

References

[1] Saichev A I and Woyczynski W A *Physica D* to appear
[2] Burgers J M 1974 *The Nonlinear Diffusion Equation* (Dordrecht: Reidel)
[3] Fournier J D and Frisch U 1983 L'équation de Burgers déterministe et statistique *J. Méc. Théor. Appl.* **2** 699–750
[4] Gurbatov S, Malakhov A and Saichev A 1991 *Nonlinear Random Waves and Turbulence in Nondispersive Media: Waves, Rays and Particles* (Manchester: Manchester University Press)
[5] Vergassola M, Dubrulles B, Frisch U and Noullez A 1994 *Astron. Astrophys.* **289** 325–56
[6] Karatzas I and Shreve S E 1991 *Brownian Motion and Stochastic Calculus* 2nd edn (Berlin: Springer)
[7] Gihman I I and Skorohod A V 1979 *The Theory of Stochastic Processes* vol III (Berlin: Springer)
[8] Garbaczewski P, Kondrat G and Olkiewicz R 1997 *Phys. Rev.* E **55** 1401–11